Advection-Diffusion Equation (ADE) in Porous Media

Diffusion = Conduction

We have discussed two physical problems

1) Flow in porous media

$$\underbrace{-\nabla\cdot[K\nabla h] = f_s}_{} \quad \text{and} \quad q = -K\nabla h$$

2) Heat conduction

$$\rho c_p \frac{\partial T}{\partial t} \underbrace{-\nabla\cdot[\kappa\nabla T] = \rho H}_{}$$

$\Rightarrow$ same : $\underline{\underline{L}} = -\underline{\underline{D}}\,\underline{\underline{Kd}}\,\underline{\underline{G}}$

Combine these problems to account for advective heat transport by pore fluid.

Need to update energy balance:

1) Account for porous medium
2) Advective heat transport

Energy Conservation Equation

Internal energy: Energy of a body not associated with Kinetic or potential energy.

Internal energy ~ thermal energy/ heat

symbol: U units: Joule $= \left[\frac{ML^2}{T^2}\right]$

specific internal energy / energy density

$u = \frac{U}{m}$ $m = \text{mass}$ $\left[\frac{L^2}{T^2} = \frac{J}{kg}\right]$

$$\boxed{du = c_p \, dT}$$

T = temperature

c_p = specific heat capacity at const. pressure $\left[\frac{J}{kg\,K} = \frac{L^2}{T^2\Theta}\right]$

Physical interpretation:

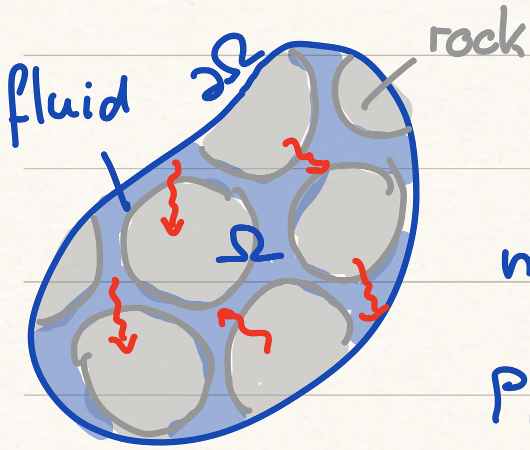
c_p is the heat required to raise the temerature of 1 kg by 1 degree K.

Energy density: $\boxed{u(T) = u_0 + c_p (T - T_0)}$

u_0 = ref. energy

T_0 = ref. temperature

Energy of rock-fluid system



Two-phase system $p \in [r, f]$

ϕ_p = volume fraction of phase p

$m_p = \rho_p V_p$ mass of phase p $[M]$

ρ_p = density of phase p $\left[\frac{M}{L^3}\right]$

$V_p = \phi_p V$ volume of phase p $[L^3]$

$\phi = \phi_f$ = porosity $\qquad V = V_f + V_r$ = total volume

Internal energy of rock:

$$U_r = u_r m_r = u_r \rho_r V_r = u_r \rho_r \phi_r V = \int_\Omega (1-\phi) \rho_r u_r \, dV$$

Similarly we have for the fluid

$$U_f = \int_\Omega \phi \rho_f u_f \, dV$$

where $\quad u_f = u_{0,f} + c_{p,f} (T_f - T_0)$

$\qquad\qquad u_r = u_{0,r} + c_{p,r} (T_r - T_0)$

choosing $u_{0,r} = u_{0,f} = T_0 = 0$

$\Rightarrow$ Internal energy of phases in Ω

$$U_r = \int_\Omega (1-\phi)\rho_r c_{p,r} T_r \, dV$$

$$U_f = \int_\Omega \phi \rho_f c_{p,f} T_f \, dV$$

Total internal energy of porous medium

$$U_T = U_r + U_f = \int_\Omega (1-\phi)\rho_r c_{p,r} T_r + \phi \rho_f c_{p,f} T_f \, dV$$

Assumption of local thermal equilibrium

$$T_f = T_r = T$$

$$\Rightarrow U_T = \int_\Omega \underbrace{[\phi \rho_f c_{p,f} + (1-\phi)\rho_r c_{p,r}] T}_{e} \, dV$$

e = total energy density of porous medium per unit volume $\left[\frac{J}{m^3} = \frac{M}{LT^2}\right]$

General balance equation: $\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} = \hat{f}_s$

u = unknown, $\underline{j}$ = flux, $\hat{f}_s$ = source/sink

1) Unknown

$$\boxed{e = (\phi \rho_f c_{p,f} + (1-\phi) \rho_r c_{p,r}) T}$$

Note: Conserved quantity ! ⇒ no source/sink term

$\overline{\rho c_p} = \phi \rho_f c_{p,f} + (1-\phi) \rho_r c_{p,r} \Rightarrow e = \overline{\rho c_p}\, T$

2) Energy fluxes

a) Conductive heat flux

Fourier's law: $\underline{j}_c = -\kappa \nabla T$

where κ = thermal conductivity

units $\left[\frac{W}{mK} = \frac{ML}{T^3 \Theta} \right]$

This applies in each phase:

$\underline{j}_{c,f} = -\kappa_f \nabla T \qquad \underline{j}_{c,s} = -\kappa_s \nabla T$

Total conductive flux

$\underline{j}_c = \phi\, \underline{j}_{c,f} + (1-\phi)\, \underline{j}_{c,s}$

$$\boxed{\underline{j}_c = -\left[\phi \kappa_f + (1-\phi)\kappa_s\right] \nabla T}$$

mean conductivity: $\bar{\kappa} = \phi \kappa_f + (1-\phi)\kappa_s$

$$\underline{j}_c = -\bar{\kappa} \nabla T$$

b) advective heat flux

$$\underline{j}_A = \underline{v} \rho u = \underline{v} \rho c_p T$$

applies in each phase

$$\underline{j}_{A,f} = \underline{v}_f \rho_f c_{p,f} T$$

$$\underline{j}_{A,s} = \underline{v}_s \rho_s c_{p,s} T$$

Total advective heat flux

$$\underline{j}_A = \phi \underline{j}_{A,f} + (1-\phi) \underline{j}_{A,s}$$

$$= \underbrace{\phi \underline{v}_f}_{q} \rho_f c_{p,f} T + (1-\phi) \cancelto{0}{\underline{v}_s} \rho_s c_{p,s} T$$

$$\Rightarrow \boxed{\underline{j}_A = q \rho_f c_{p,f} T}$$

3) Source/Sink $\hat{f}_s = 0$

because e is a conserved quantity

Substitute into the general balance law

$$\boxed{\overline{\rho c_p}\,\frac{\partial T}{\partial t} + \nabla \cdot \left[\underline{q}\,\rho_f c_{p,f}\, T - \bar{\kappa} \nabla T \right] = 0}$$

If $\overline{\rho c_p}$ = const.

$$\Rightarrow \quad \frac{\partial T}{\partial t} + \nabla \cdot \left[\underline{v}_e T - \bar{\alpha}\, \nabla T \right] = 0$$

$$\bar{\alpha} = \frac{\bar{\kappa}}{\overline{\rho c_p}}$$ mean thermal diffusivity

$$\underline{v}_e = \underline{v}_f \underbrace{\frac{\phi \rho_f c_{pf}}{\overline{\rho c_p}}}_{\leq 1}$$

effective velocity of thermal fronts which is less than the fluid velocity because of heat exchange with solid