Steady Advection with heating

$$\frac{\partial T}{\partial t} = 0$$

$$\Rightarrow \quad \nabla \cdot [\underline{v}_e T] = \frac{(1-\phi)\rho_s H}{\overline{\rho c_p}}$$

In 1D: $$v_e \frac{\partial T}{\partial x} = \frac{(1-\phi)\rho_s H}{\overline{\rho c_p}} \qquad x \in [0, L]$$

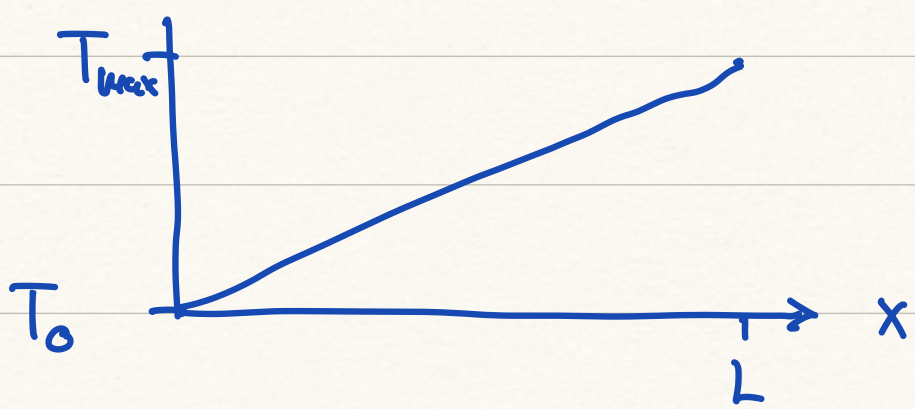
BC: $T(x=0) = T_0$

$$\frac{\partial T}{\partial x} = \frac{(1-\phi)\rho_s H}{\overline{\rho c_p}\, v_e}$$

Integrate:

$$\boxed{T = T_0 + \frac{(1-\phi)\rho_s H}{\overline{\rho c_p}\, v_e}\, x}$$

$$T_{max}: \quad T(L) = T_0 + \underbrace{\frac{(1-\phi)\rho_s H L}{\overline{\rho c_p}\, v_e}}_{\Delta T}$$



Discretization of Steady Advection

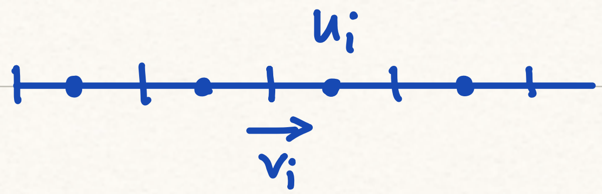
Continuous: $\nabla \cdot [\underline{v}_e T] = \frac{(1-\phi)\rho_s H}{\rho c_p}$

$\nabla \cdot \underline{a} = 1 \qquad \underline{a} = \underline{v}_e T = \text{adv. flux}$

Discrete: $\underline{\underline{D}}\ \underline{a} = \underline{f_s}$

$\underline{a}$ = discrete advective flux vector

How to compute $\underline{a}$ from $\underline{v}$ and $\underline{u}$?



$\underline{v}$ = Nf by 1 on faces

$\underline{u}$ = N by 1 in cells

Advection matrix: $\underline{a} = \underline{\underline{A}}\ \underline{u}$

$\qquad\qquad\qquad\quad Nf \cdot 1 \quad Nf \cdot N \quad N \cdot 1$

$\underline{\underline{A}}$ is a Nf by N matrix that computes $\underline{a}$ from $\underline{u}$. Shape is same as $\underline{\underline{G}}$

$\underline{\underline{A}} = \underline{\underline{A}}(\underline{v})$ must be a function of $\underline{v}$!

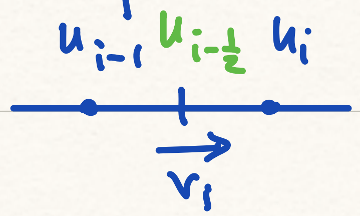
In that case we discretize:

$$\nabla \cdot [\underline{v}_e T] = 1$$

$$\underline{\underline{D}}\ \underline{\underline{A}}(\underline{v})\ \underline{u} = \underline{f_s} \qquad \underline{\underline{L}}\ \underline{u} = \underline{f_s} \qquad \underline{\underline{L}} = \underline{\underline{D}}\ \underline{\underline{A}}(\underline{v})$$

Construction of $\underline{\underline{A}}$

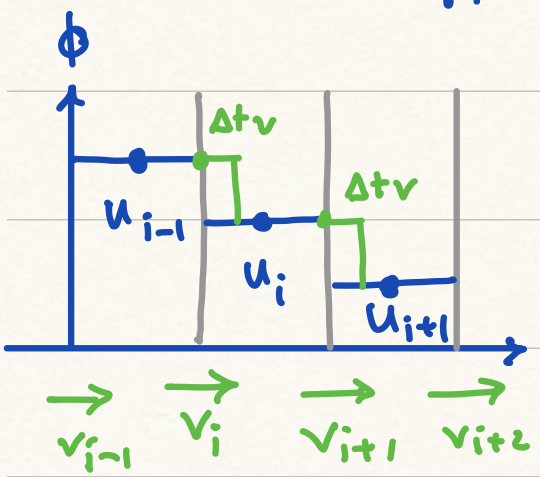
The purpose of $\underline{\underline{A}}$ is to estimate ϕ on the cell faces and to multiply by $\underline{v}$

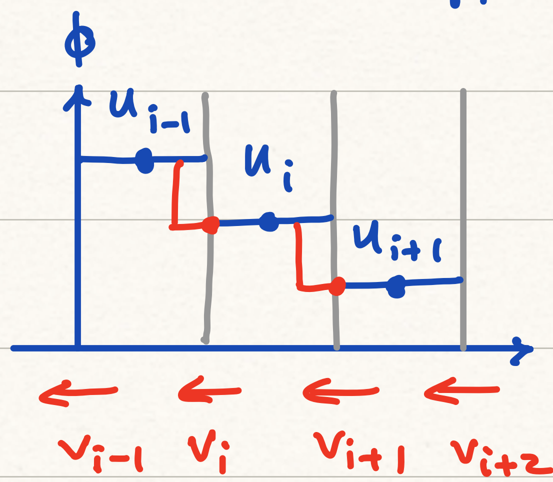


$$a_i = v_i \, u_{i-\frac{1}{2}}$$

How do we approximate $\phi_{i-1/2}$?

pp





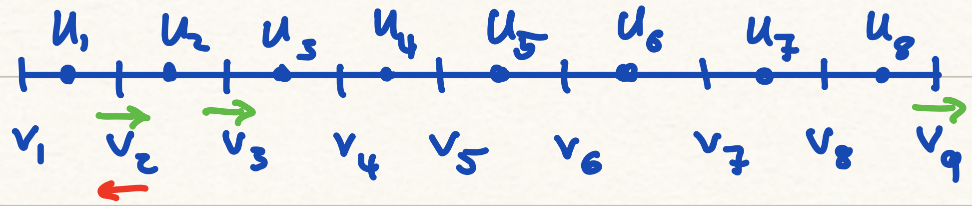
all $v_i > 0$

$v_i > 0$: $u_{i-1/2} = u_{i-1}$ $\qquad$ $v_i < 0$: $u_{i-1/2} = u_i$

From the analytic solution we know $u_{i-1/2}$ depends only on upstream/upwind ϕ values

Natural choice : $\boxed{u_{i-\frac{1}{2}} = \begin{cases} u_{i-1} & v \geq 0 \\ u_i & v < 0 \end{cases}}$

Construction of $\underline{\underline{A}}$



$$\underline{\underline{A}}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ \\ \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} v_1 & & & & & & & \\ v_2 & v_2 & & & & & & \\ & v_3 & v_3 & & & & & \\ & & v_4 & v_4 & & & & \\ & & & v_5 & v_5 & & & \\ & & & & v_6 & v_6 & & \\ & & & & & v_7 & v_7 & \\ & & & & & & v_8 & v_8 \\ & & & & & & & v_9 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \\ \\ \\ u_8 \end{pmatrix}$$

$a_2 = v_2 u_1$

$a_3 = v_3 u_2$

$a_1 = v_1 u_1$

$$\underline{\underline{A}} = \underline{\underline{A}}^{+} + \underline{\underline{A}}^{-}$$

To build $\underline{\underline{A}}$ we need to select appropriate rows of $\underline{\underline{A}}^{+}$ and $\underline{\underline{A}}^{-}$ according to sign of the corresponding entry of $\underline{v}$.

Bulid pos. and neg. velocity vectors:

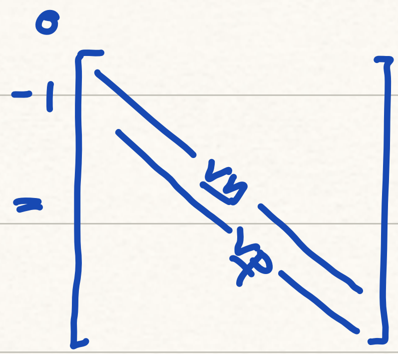
$\underline{vn} = \min(\underline{v}(1:Nx), 0)$

$\underline{vp} = \max(\underline{v}(2:Nx+1), 0)$

$$\underline{v} = \begin{bmatrix} 4 \\ -7 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix} \qquad \underline{vn} = \begin{bmatrix} 0 \\ -7 \\ 0 \\ -1 \\ -2 \\ 0 \end{bmatrix} \qquad \underline{vp} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

given these two vectors we build $\underline{\underline{A}}(\underline{v})$ as

$\underline{\underline{A}} =$



assemble with spdiags