# Advective Heat Transport

So far we have considerd heat conduction

$$\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot \kappa \nabla T = \rho H$$

In lectur 2 we derived general balance law

$$\frac{\partial u}{\partial t} + \nabla \cdot \underline{j} = f_s$$

$u = \rho c_p (T - T_0)$ is unknow heat

$\underline{j} = -\kappa \nabla T = \underline{j}_D$ is conductive heat flux $= \underline{j}_D$

$f_s = \rho H$ is radiogenic heat production

Multiple fluxes and source terms

$$\frac{\partial u}{\partial t} + \nabla \cdot \sum_{i=1}^{N_f} \underline{j}_i = \sum_{i=1}^{N_s} f_i$$

$N_f = \#$ of fluxes

$N_s = \#$ of sources

Today add advective flux

general: $\boxed{\underline{j}_A = \underline{v}\, u}$ $\underline{v}$ = velocity $\left[\frac{L}{T}\right]$

heat: $\boxed{\underline{j}_A = \underline{v} \rho c_p (T - T_0)}$

Heat flux = Thermal Energy Flux $= \frac{J}{L^2 T} = \frac{M\cancel{L^2}}{T^2}\frac{1}{\cancel{L^2}T} = \frac{M}{T^3}$

Energy: $J = Nm = \frac{ML}{T^2}L = \frac{ML^2}{T^2}$

check units of advective heat flux

$\underline{v} = [\frac{L}{T}]$ $\rho = [\frac{M}{L^3}]$ $c_p = \left[\frac{J}{kg\,K} = \frac{L^2}{T^2\Theta}\right]$ $T = [\Theta]$

$\underline{j}_A = \frac{L}{T}\frac{M}{\cancel{L^3}}\frac{\cancel{L^2}}{T^2\cancel{\Theta}}\cancel{\Theta} = \frac{M}{T^3}$ ✓

Adding advective flux to energy balance

$$\frac{\partial u}{\partial t} + \nabla\cdot[\underline{j}_A + \underline{j}_D] = f_s$$

$$\Rightarrow \quad \rho c_p \frac{\partial T}{\partial t} + \nabla\cdot[\underline{v}\rho c_p (T - T_0) - \kappa\nabla T] = \rho H$$

$$\nabla\cdot[\underline{v}\rho c_p T_0] = \rho c_p T_0\,\nabla\cdot\underline{v} = 0$$

Energy balance: $\boxed{\rho c_p \frac{\partial T}{\partial t} + \nabla\cdot[\underline{v}\rho c_p T - \kappa\nabla T] = \rho H}$

if $\kappa = \rho H = 0$ and $\rho c_p = \text{const.}$

$$\frac{\partial T}{\partial t} + \nabla\cdot[\underline{v}T] = 0$$

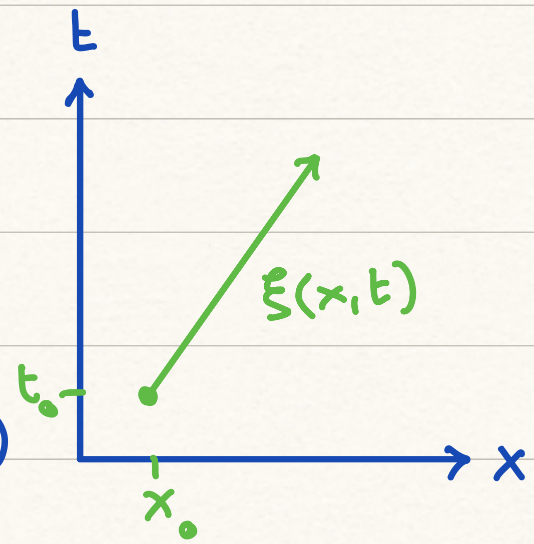
$$\boxed{\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T = 0}$$

# Solving Advection with Method of Characteristics

Consider 1D heat transport

PDE: $\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial x} = 0 \qquad x \in \mathbb{R}$

IC: $T(x,0) = T_0(x)$




Idea: Find a coordinate $\xi(x,t)$ along which PDE reduces to an ODE. $T(x,t) \equiv \theta(\xi(x,t))$

$\xi \equiv$ characteristic curve

Total change of T along characteristic:

$$\left.\begin{aligned} \frac{d\theta}{d\xi} &= \frac{\partial T}{\partial t}\frac{dt}{d\xi} + \frac{\partial T}{\partial x}\frac{dx}{d\xi} \\ \text{PDE: } & \frac{\partial T}{\partial t} \qquad + v \frac{\partial T}{\partial x} = 0 \end{aligned}\right\} \xrightarrow{\text{comparing}} \begin{aligned} &1)\ \frac{d\theta}{d\xi} = 0 \\ &\left.\begin{aligned} &2)\ \frac{dt}{d\xi} = 1 \\ &3)\ \frac{dx}{d\xi} = v \end{aligned}\right\} \frac{dx}{dt} = v \end{aligned}$$

Solve for char. eqn: $x - x_0 = v(t - t_0) \qquad t_0 = 0$

$t_0 = 0: \quad x_0 = \underbrace{x - vt}_{\text{travelling wave coord.}}$

Substitute into IC

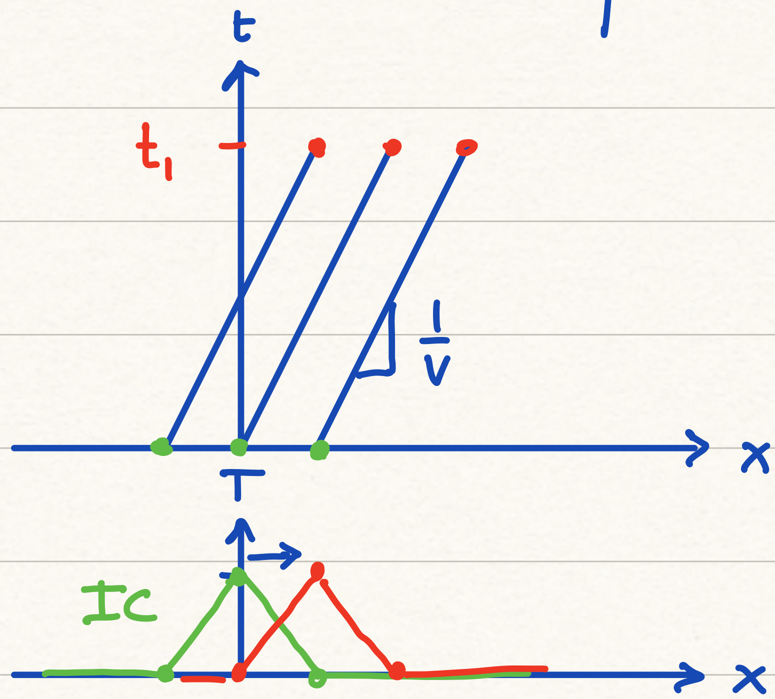
$$T(x, t=0) = T_0(x_0) = T_0(x - vt)$$

General Solution to linear Advection Equ

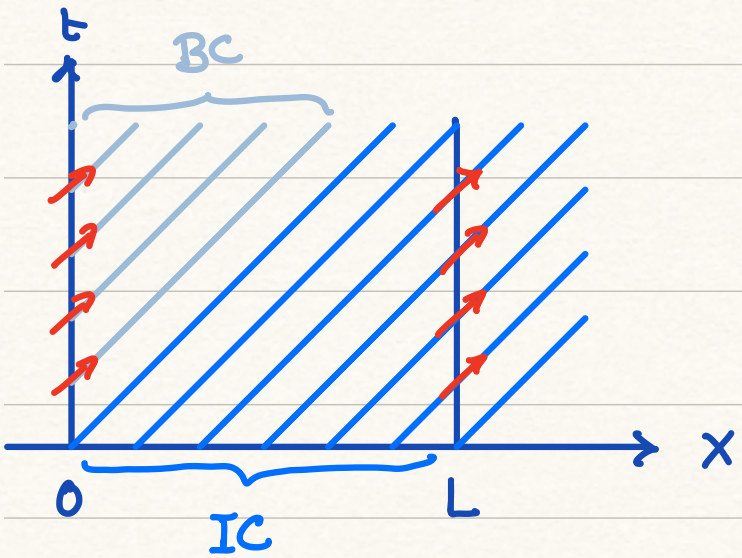
$$\boxed{T(x,t) = T_0(x-vt)}$$

IC is shifted to the right ($v>0$) without change in shape!

$\Rightarrow$ graphical interp.




Now consider a finite domain




- don't need outflow BC
- need BC on inflow side

$\Rightarrow$ GEO 366M Math. Methods

$\Rightarrow$ use in discretization

# Steady Advection

Example: Adiabatic mantle upwelling

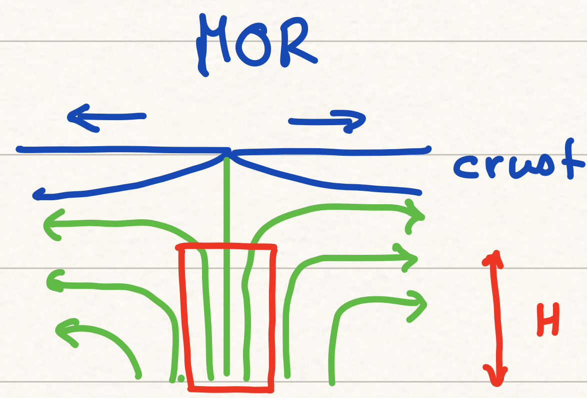
→ Mid-Ocean Ridge (MOR)

Corner flow in mantle

⇒ steady advection

Decompression cools mantle

⇒ sink term




$$\nabla \cdot [\underline{v} \rho c_p T] = -\alpha \rho g T \, \underline{v} \cdot \hat{z}$$

$\rho c_p = \text{const}$

$$\boxed{\nabla \cdot [\underline{v} T] = -\frac{\alpha g T}{c_p} \underline{v} \cdot \hat{z}}$$

$\alpha$ = thermal expansion coef.

$= 4 \cdot 10^{-5} \frac{1}{K}$

(GEO 325C Cont. Mech.)

Directly beneath ridge $\underline{v}$ is approx. vertical

$$\frac{d}{dz}[vT] = -\frac{\alpha g T v}{c_p}$$

if $v = \text{const.}$

$$\Rightarrow \boxed{\frac{dT}{dz} = -\frac{\alpha g}{c_p} T}$$

adiabatic $T$ gradient

$\alpha = 4 \cdot 10^{-5}$ $\quad T = 1500\ K$

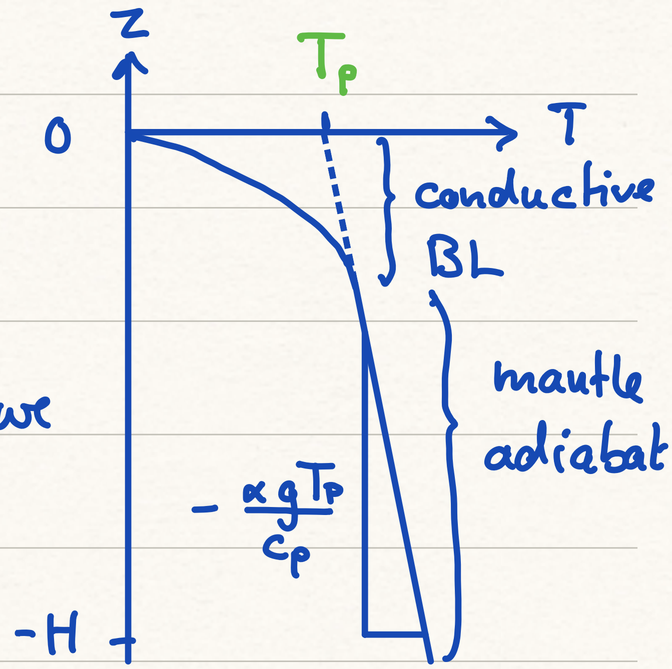
$c_p = 1260 \ \frac{J}{kg\,K}$

$g = 9.81 \ \frac{m}{s^2}$

$$\frac{dT}{dz} \approx 5 \cdot 10^{-4} \frac{K}{m}$$

$$\approx 0.5 \frac{K}{km}$$

To solve we need BC, but inflow bnd is arbitrary




$\Rightarrow$ mantle potential temperature = adiabat at surface $=: T_p$

ODE: $\quad \frac{dT}{dz} = -\frac{\alpha g}{c_p} T, \quad z \in [-H, 0]$

BC: $\quad T(z=0) = T_p$

Integrate: $\quad T = a \exp\left(-\frac{\alpha g}{c_p} z\right)$

$T(z) = a\, e^0 = T_p$

Mantle adiabat:

$$\boxed{T(z) = T_p\, e^{-\frac{\alpha g}{c_p} z}}$$

$$\frac{\alpha g}{c_p} \approx \frac{5 \cdot 10^{-4}\ 10}{10^3} = 5 \cdot 10^{-6}\ \frac{1}{m} \ll 1$$

For $|z| \ll 10^6 m = 10^3 km$: $\quad \boxed{T(z) \approx \underbrace{-\frac{\alpha g T_p}{c_p}}_{\text{adiabatic gradient}} z}$