Amplification matrix

General discretization with θ-method

$$\underline{\underline{M}}(\underline{h}^{n+1} - \underline{h}^{n}) + \Delta t \, \underline{\underline{L}} \, \underline{h}^{?} = \Delta t \, \underline{f_s}$$

Collecting terms

$$\underline{\underline{Im}} \, \underline{h}^{n+1} = \underline{\underline{Ex}} \, \underline{h}^{n} + \Delta t \, \underline{f_s}$$

In absence of BC (other than natural) and source/sink terms we expect solution of diffusion problem to decay.

for $f_s = 0$ we can write:

$$\underline{h}^{n+1} = \underbrace{\underline{\underline{Im}}^{-1} \, \underline{\underline{Ex}}}_{\underline{\underline{A}} = \text{amplification matrix}} \, \underline{h}^{n} = \underline{\underline{A}} \, \underline{h}^{n}$$

So that:

$$\underline{h}^{n+1} = \underline{\underline{A}} \, \underline{h}^{n} = \underline{\underline{A}} \, \underline{\underline{A}} \, \underline{h}^{n-1} = \underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}} \, h^{n-2} = \ldots = \underline{\underline{A}}^{n} \, \underline{h}^{0}$$

where $\underline{h}^{0}$ is the initial condition.

To evolve the solution we just keep multiplying by $\underline{\underline{A}}$

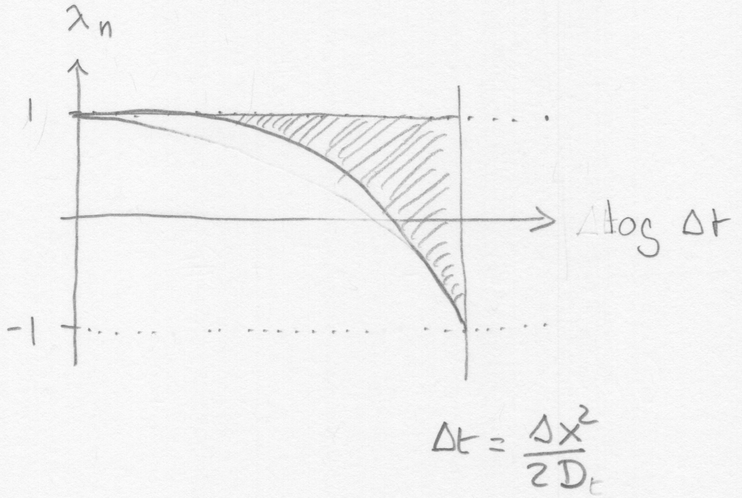
To evaluate the matrix exponential $\underline{\underline{A}}^{n}$ we use the spectral decomposition $\underline{\underline{A}} = \underline{\underline{Q}} \, \underline{\underline{\Lambda}} \, \underline{\underline{Q}}^{-1}$

where $\underline{\underline{Q}}$ is a square matrix of eigenvectors (as columns) and $\underline{\underline{\Lambda}}$ is a diagonal matrix of eigenvalues.

Since $\underline{\underline{Q}} \, \underline{\underline{Q}}^{-1} = I \Rightarrow \underline{\underline{A}}^{n} = \underline{\underline{Q}} \, \underline{\underline{\Lambda}}^{n} \, \underline{\underline{Q}}^{-1}$

The solution decays if $\max|\lambda_n| < 1$

Forward Euler ($\theta = 1$):



Physical interpretation

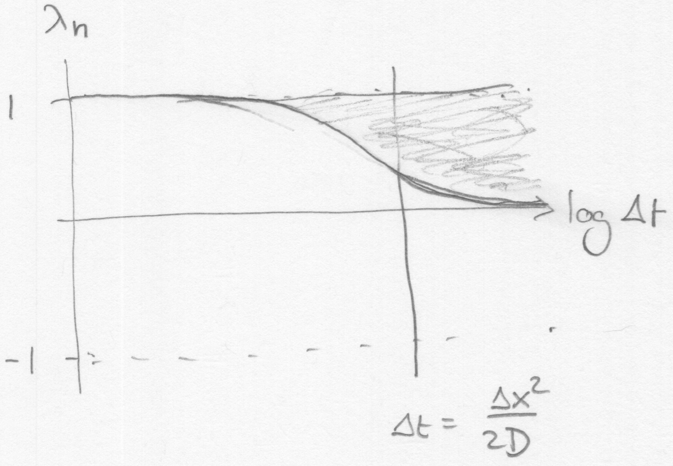
diffusion front $x_f \sim \sqrt{4Dt}$

time required to diffuse across one grid cell $\Delta x \sim \sqrt{4D\Delta t}$

$\Rightarrow \Delta t \sim \frac{\Delta x^2}{4D}$

We cannot take time steps larger than the time it takes to diffuse across a cell

Backward Euler



Crank-Nicholson