Dirichlet BC's & Constraints

Danube-Tisza Interfluve

PDE : $-\nabla\cdot[bK\nabla h] = q_p$ $\quad x \in [0, L]$

BC: $h(0) = h_D \quad h(L) = h_T$

$\Rightarrow$ heterogeneous (non-zero) Dirichlet BC's

$K = \text{const.}$ & $b = \text{const.}$ $\Rightarrow$ PDE: $-\nabla^2 h = \frac{q_p}{bK} = f_s$

Initially lets solve homogeneous problem

$$h(0) = h(L) = 0$$

not realistic but helpful step

Discretize PDE:

$$\underbrace{-\underline{\underline{D}}\,\underline{\underline{G}}}_{\underline{\underline{L}}}\,\underline{h} = \underline{f_s} \qquad \underline{f_s} = \frac{q_p}{bK}\,\text{ones}(N,1)$$

$$\Rightarrow \quad \boxed{\underline{\underline{L}}\,\underline{h} = \underline{f_s}}$$

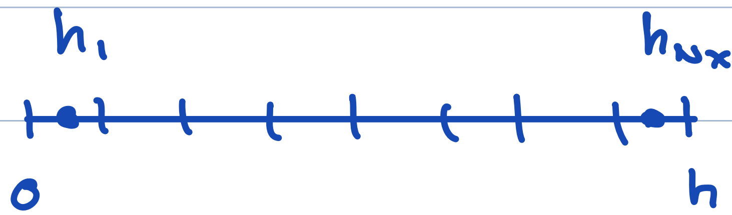
$\underline{\underline{L}} = -\underline{\underline{D}}\,\underline{\underline{G}}$ "system matrix"

Matlab note: Use backslash to solve lin. sys!

$\underline{h} = \underline{\underline{L}} \backslash \underline{f_s}$ (don't invert $\underline{\underline{L}}$)

Homogeneous BC's

$h(0) = h(L) = 0$



$\Rightarrow h_1 = 0 \; , \; h_{Nx} = 0$

note: the BC are imposed at cell centers

$\Rightarrow$ introduces an error

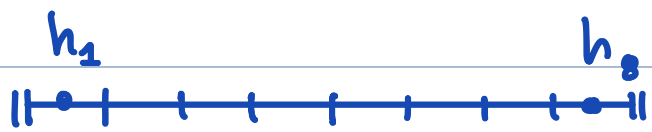
(in 1D we can just shift grid!)

BC's are Linear $\longrightarrow$ write as linear system

$$\underset{N_c \cdot N_x}{\underline{\underline{B}}} * \underset{N_x \cdot 1}{\underline{h}} = \underset{N_c \cdot 1}{\underline{0}}$$

$\underline{\underline{B}}$ = "constraint matrix

N_c = number of constraints



$$\underbrace{N_c \begin{bmatrix} 1 & & & & & & & \\ & & & & & & & 1 \end{bmatrix}}_{N_x}^{\underline{\underline{B}}} \quad \overset{\underline{h}}{\begin{bmatrix} h_1 \\ \\ \\ \\ \\ \\ \\ h_8 \end{bmatrix}} \; N_x \; (1) \quad = \quad \overset{\underline{0}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \; N_c \; (1)$$

Full discrete problem

$$\boxed{\begin{array}{ll} \text{PDE:} & \underline{\underline{L}}\,\underline{h} = \underline{f_s} \\ \text{BC:} & \underline{\underline{B}}\,\underline{h} = \underline{0} \end{array}}$$

$\underline{\underline{L}} = N_x \cdot N_x$ system matrix

$\underline{\underline{B}} = N_c \cdot N_x$ constraint matrix

Neither system has a unique solution but together they do!

$\Rightarrow$ combine systems by eliminating constraints in $\underline{\underline{B}}$ from $\underline{\underline{L}}$.

Reduced Linear System

Constraints reduce number of unknowns to $N_x - N_c$

$\Rightarrow$ solve smaller system

reduced system: $\boxed{\underline{\underline{L_r}}\,\underline{h_r} = \underline{f_{sr}}}$

$\underline{h_r}$ is $N_x - N_c \cdot 1$ reduced solution vector

$\underline{f_{sr}}$ is $N_x - N_c \cdot 1$ reduced r.h.s. vector

$\underline{\underline{L_r}}$ is $(N_x - N_c) \cdot (N_x - N_c)$ reduced system matrix

What is relation between: $\underline{h}r \longleftrightarrow \underline{h}$

$\underline{f}sr \longleftrightarrow \underline{f}s$

$\underline{\underline{L}}r \longleftrightarrow \underline{\underline{L}}$

Projection matrix

Two vectors of different length are related by a rectangular matrix.

$$\underset{Nx \cdot 1}{\underline{h}} = \underset{Nx \cdot (Nx - Nc)}{\underline{\underline{N}}} * \underset{(Nx - Nc) \cdot 1}{\underline{h}r}$$

What is $\underline{\underline{N}}$?

For now just require $\underline{\underline{N}}$ is orthonormal.

$$\underline{\underline{N}} = \begin{bmatrix} | & | & & | \\ \underline{n}_1 & \underline{n}_2 & \cdots & \underline{n}_{Nx-Nc} \\ | & | & & | \end{bmatrix}$$

$\underline{n}_i$ is the i-th column

$\underline{n}_i \cdot \underline{n}_i = 1$ (normal)

$\underline{n}_i \cdot \underline{n}_{j \neq i} = 0$ (ortho)

It follows:

a) $$\underline{\underline{N}}^T \quad \underline{\underline{N}} \quad = \quad \underline{\underline{I}}_r$$

$(N_x - N_c) \cdot N_x \quad N_x \cdot (N_x - N_c) \quad (N_x - N_c) \cdot (N_x - N_c)$

b) $$\underline{\underline{N}} \quad \underline{\underline{N}}^T \quad = \quad \underline{\underline{I}}'$$

$N_x \cdot (N_x - N_c) \quad (N_x - N_c) \cdot N_x \quad N_x \cdot N_x$

$\underline{\underline{I}}_r$ identity in reduced space

$\underline{\underline{I}}'$ "identity in full space but with N_c zeros on the diagonal

if $$\underline{u} = \underline{\underline{N}} \, \underline{u_r}$$

$$\underline{\underline{N}}^T \underline{u} = \underline{\underline{N}}^T \underline{\underline{N}} \, \underline{u_r} = \underline{\underline{I}}_r \underline{u_r} = \underline{u_r}$$

$$\Rightarrow \underline{u_r} = \underline{\underline{N}}^T \underline{u}$$

$\underline{\underline{N}}$ allows transfer between full & reduced space

$$\boxed{\begin{aligned} \underline{u} &= \underline{\underline{N}} \, \underline{u_r} \\ \underline{u_r} &= \underline{\underline{N}}^T \underline{u} \end{aligned}}$$

we say that $\underline{\underline{N}}^T$ projects $\underline{u}$ into reduced space.

Similarly: $$\underline{f_s} = \underline{\underline{N}} \, \underline{f_{sr}}$$

$$\underline{f_{sr}} = \underline{\underline{N}}^T \underline{f_s}$$

How is $\underline{\underline{L}}$ projected into reduced space?

$$\underline{\underline{L}}\,\underline{h} = \underline{fs} \longrightarrow \underline{\underline{Lr}}\,\underline{ur} = \underline{fsr}$$

already know $\underline{u} = \underline{\underline{N}}\,\underline{ur}$ and $\underline{fs} = \underline{\underline{N}}\,\underline{fsr}$

$$\Rightarrow \underline{\underline{L}}\,\underline{\underline{N}}\,\underline{hr} = \underline{\underline{N}}\,\underline{fsr}$$

need to remove $\underline{\underline{N}}$ on l.h.s.

$$\underline{\underline{N}}^T\underline{\underline{L}}\,\underline{\underline{N}}\,\underline{hr} = \underbrace{\underline{\underline{N}}^T\underline{\underline{N}}}_{\underline{\underline{Ir}}}\,\underline{fsr} = \underline{fsr}$$

$$\underbrace{\underline{\underline{N}}^T\underline{\underline{L}}\,\underline{\underline{N}}}_{\underline{\underline{Lr}}}\,\underline{hr} = \underline{fsr}$$

Hence we have:

$\underline{\underline{Lr}}\,\underline{hr} = \underline{fsr}$ where

$$\boxed{\begin{aligned} \underline{\underline{Lr}} &= \underline{\underline{N}}^T\underline{\underline{L}}\,\underline{\underline{N}} \\ \underline{hr} &= \underline{\underline{N}}^T\underline{h} \\ \underline{fsr} &= \underline{\underline{N}}^T\underline{fs} \end{aligned}}$$

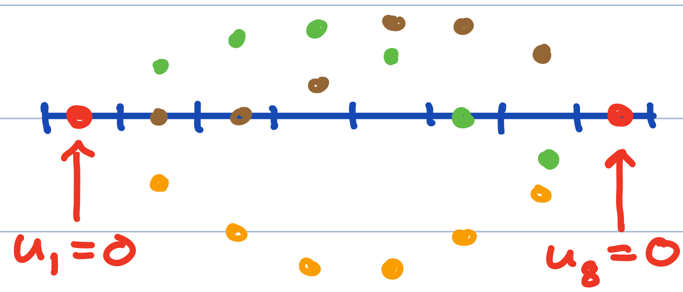
Just need to find $\underline{\underline{N}}$!

$\underline{\underline{N}}$ needs to contain information about BC's.

$\Rightarrow$ related to $\underline{\underline{B}}$

Constraint System: $\underline{\underline{B}}\,\underline{h} = \underline{0}$

$\Rightarrow$ many solutions



- soln 1: $\underline{h}_1$
- soln 2: $\underline{h}_2$
- soln 3: $\underline{h}_3$

$\Big\}$ Nullspace of $\underline{\underline{B}}$: $\mathcal{N}(\underline{\underline{B}})$

$\Rightarrow$ search for solutions in $\mathcal{N}(\underline{\underline{B}})$!

$\mathcal{N}(\underline{\underline{B}})$ is the reduced space

Basis of $\mathcal{N}(\underline{\underline{B}})$: set of linearly independent vectors $\{\underline{n}_i\}$ that span the space.

$$\underline{h}_1 = \sum \alpha_i \underline{n}_i \qquad \underline{h}_2 = \sum \beta_i \underline{n}_i \qquad \underline{h}_3 = \sum \gamma_i \underline{n}_i$$

$\Rightarrow$ $\underline{h} = \sum \delta_i \underline{n}_i$ weights $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$

Matrix $\underline{\underline{N}}$ has basis vectors as columns

$$\underline{\underline{N}} = \begin{bmatrix} | & | & | & & | \\ \underline{n}_1 & \underline{n}_2 & \underline{n}_3 & \cdots & \underline{n}_{Nx-Nc} \\ | & | & | & & | \end{bmatrix}$$

$$\underline{h} = \sum \delta_i \underline{n}_i = \underline{\underline{N}}\,\underline{\delta} \qquad \underline{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \vdots \\ \delta_{Nx-Nc} \end{bmatrix}$$

$\underline{h} = \underline{\underline{N}}\,\underline{hr}$ $\Rightarrow$ $\underline{hr} = \underline{\delta}$!

Note: basis $\{\underline{n}_i\}$ is not unique !

simplest orthonormal basis will do

How do we find $\{\underline{n}_i\}$ and hence $\underline{\underline{N}}$?

In Matlab : $\underline{\underline{N}}$ = null ($\underline{\underline{B}}$)

$\underline{\underline{N}}$ = spnull ($\underline{\underline{B}}$)

→ download Matlab Central !

Problem: General but expensive.

Simple constraints → $\underline{\underline{N}}$ easy to find

$$\underline{\underline{B}} = \begin{bmatrix} 1 & & & & & & & \\ & & & & & & & 1 \end{bmatrix}$$

$$\underline{\underline{I}} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ \underline{\underline{N}} \end{array}$$

$$\begin{array}{c|c} & N_x - N_c \\ \hline N_x & \begin{bmatrix} & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ & & & & & \end{bmatrix} \\ & \underline{n}_1 \;\; \underline{n}_2 \;\; \cdot \;\; \cdot \;\; \cdot \;\; \underline{n}_6 \end{array} = \underline{\underline{N}}$$

$\Rightarrow$ spliting $\underline{\underline{I}}$ into $\underline{\underline{B}}$ & $\underline{\underline{N}}$

$\underline{n}_i$'s are the columns of $\underline{\underline{I}}$ not in $\underline{\underline{B}}$

since each $\underline{n}_i$ has only one non-zero

entry for one unknowns $\Rightarrow$ orthonormal

$\Rightarrow$ simplest possible basis for $\mathcal{N}(\underline{\underline{B}})$

In Matlab:

1) Vector containing Dirichlet cells:

BC.dof_dir = [Grid.dof_xmin; Grid.dof_xmax];

2) Build $\underline{\underline{B}}$ selecting rows from $\underline{\underline{I}}$

$\underline{\underline{B}}$ = $\underline{\underline{I}}$(BC.dof_dir, :);

3) Build $\underline{\underline{N}}$ deleting columns from $\underline{\underline{I}}$

$\underline{\underline{N}}$ = $\underline{\underline{I}}$;

$\underline{\underline{N}}$(:, BC.dof_dir) = [];

Heterogeneous BC's

Return to original geotherm problem

PDE: $-\nabla \cdot K \nabla h = q_p \qquad x \in [0, L]$

BC: $h(0) = h_D \qquad h(L) = h_T$

Discrete systems

$$\boxed{\begin{aligned} \underline{\underline{L}}\,\underline{h} &= f_s \\ \underline{\underline{B}}\,\underline{h} &= g \end{aligned}} \qquad g = \begin{bmatrix} h_D \\ h_T \end{bmatrix}$$

$\underline{\underline{B}}$ same as before → location of BC's

Because $\underline{\underline{B}}\,\underline{u} = g$ is linear

⇒ decompose: $\underline{u} = \underline{u}_0 + \underline{u}_p$

homogeneous soln.: $\underline{\underline{B}}\,\underline{u}_0 = \underline{0}$

particular soln.: $\underline{\underline{B}}\,\underline{u}_p = g$

$$\left.\begin{array}{l} \underline{\underline{B}}\,\underline{u}_0 = \underline{0} \\ \underline{\underline{B}}\,\underline{u}_p = g \end{array}\right\} \underline{\underline{B}}\,(\underbrace{\underline{u}_0 + \underline{u}_p}_{\underline{u}}) = g$$

Note: $\underline{u}$ is unique (assuming suitable BC's)

split into $\underline{u}_0 + \underline{u}_p$ is not unique

→ simplest (obvious) choice

Two questions: 1) How do we find $\underline{u}_p$?

2) Given $\underline{u}_p$ how do we find associated $\underline{u}_0$?

Start with Q2: Suppose we know $\underline{u}_p$

$\underline{\underline{L}}\,(\underline{u}_0 + \underline{u}_p) = \underline{f_s}$ $\qquad$ $\underline{u}_p$ is known → r.h.s

$$\underline{\underline{L}}\,\underline{u}_0 = \underline{f_s} \underbrace{- \underline{\underline{L}}\,\underline{u}_p}_{\underline{f_d}\,=\,\text{Dirichlet r.h.s.}} \qquad \underline{\underline{L}}\,\underline{u}_0 = \underline{f_s} + \underline{f_d} \qquad \underline{f_d} = -\underline{\underline{L}}\,\underline{u}_p$$

Form reduced system as before:

$$\underline{\underline{L}}\,\underline{u}_{0r} = \underline{f}_r \qquad \underline{f}_r = \underline{\underline{N}}^T(\underline{f}_s + \underline{f}_d)$$

where $\underline{u}_{0r}$ is reduced homogeneous solution

$$\underline{u}_0 = \underline{\underline{N}}\,\underline{u}_{0r}$$

$$\Rightarrow \quad \underline{u} = \underline{u}_0 + \underline{u}_p \qquad \text{full solution}$$

Q1: How do we find u_p?

$\underline{u}_p$ is not unique: $\underline{\underline{B}}\,\underline{u}_p = g$

$\underline{\underline{L}}\,\underline{u}_p \neq \underline{f}_s$ does no satisfy PDE by itself

$\Rightarrow$ just satisfy $\underline{\underline{B}}\,\underline{u}_p = g$!

Simplest solution: $\underline{u}_p = \begin{bmatrix} u_D \\ 0 \\ 0 \\ \vdots \\ 0 \\ u_T \end{bmatrix}$

$\underline{\underline{B}} \rightarrow$ Location & $g \rightarrow$ value

Matlab: hp(dof_dir) = g

Simple because our constraints are simple.

General approach for arbitrary constraints

$\Rightarrow$ solve reduced system for $\underline{h_p}$

should be a $N_c \cdot N_c$ system!

$$\underset{N_c\cdot 1}{\underline{h_{pr}}} = \underset{N_c\cdot N_x}{\underline{\underline{B}}}\ \underset{N_x\cdot 1}{\underline{h_p}} \quad \rightarrow \quad \underset{N_x\cdot 1}{\underline{h_p}} = \underset{N_x\cdot N_c}{\underline{\underline{B}}^T}\ \underset{N_c\cdot 1}{\underline{h_{pr}}}$$

substitute:

$$\underline{\underline{B}}\ \underline{h_p} = g$$

$$\underset{N_c\cdot N_x}{\underline{\underline{B}}}\ \underset{N_x\cdot N_c}{\underline{\underline{B}}^T}\ \underline{h_{pr}} = g \qquad \underset{N_c\cdot N_c}{\underline{\underline{B}}\,\underline{\underline{B}}^T}$$

Particular reduced system: $\boxed{\underline{\underline{B}}\,\underline{\underline{B}}^T \underline{h_r} = g}$

Summary of Dirchlet BC implementation

Two new functions:

1) [B, N, fn] = build_bnd (BC, Grid, I)

2) h = solve_Lbvp(L, f, B, g, N)

$\Rightarrow$ next home work

General implementation

works for 1D, 2D, 3D, cylindrical, spherical, ...

Inside solve_lbvp.m

1) Find $\underline{h}_p$

$$\underline{\underline{B}}\,\underline{\underline{B}}^T \underline{h}_{pr} = g \longrightarrow \underline{h}_p = \underline{\underline{B}}^T \underline{h}_{pr}$$

2) Find associated homogeneous solution

$$\underline{\underline{L}}_r \underline{h}_{or} = \underline{f}_r \longrightarrow \underline{h}_o = \underline{\underline{N}}\,\underline{h}_{or}$$

3) Full solution: $\underline{h} = \underline{h}_o + \underline{h}_p$