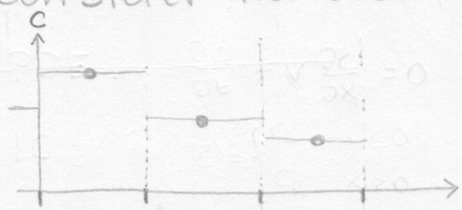
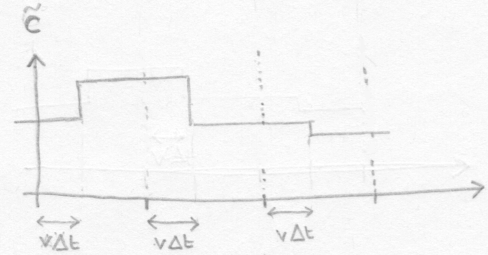
Origin of the timestep restriction

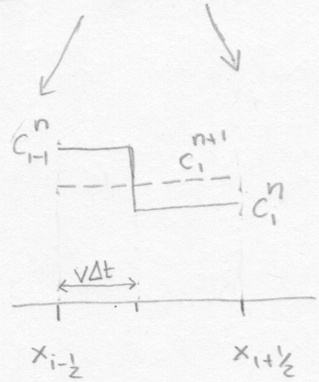
Consider the evolution of the discrete solution



The value at the cell centers represents the average over the cell. Discrete solution is a series of steps at cell faces



These steps move with the advection velocity v. So after a time step Δt steps have moved by $v\Delta t$.



The new concentration at the end of the timestep is the average

$$c_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{c}\, dx = \frac{1}{\Delta x}\left(\int_{x_{i-\frac{1}{2}}}^{x_{-\frac{1}{2}}+v\Delta t} c_{i-1}^n\, dx + \int_{x_{i-\frac{1}{2}}+v\Delta t}^{x_{+\frac{1}{2}}} c_i^n\, dx\right)$$

$$= \frac{1}{\Delta x}\left(c_{i-1}^n\, v\Delta t + c_i^n(\Delta x - v\Delta t)\right)$$

$$\boxed{c_i^{n+1} = \frac{v\Delta t}{\Delta x}\, c_{i-1}^n + \left(-\frac{v\Delta t}{\Delta x}\right) c^n}$$

Note $\frac{v\Delta t}{\Delta x} = \alpha$ CFL number

Interpretation $c^{n+1} = \alpha\, c_{i-1}^n + (1-\alpha)\, c_i^n$

- weighted average / interpolation of previous values
- if $\alpha = \frac{v\Delta t}{\Delta x} > 1$ interpolation becomes an extrapolation

because front has swept through entire cell

Stability condition. $\boxed{\alpha = \frac{v\Delta t}{\Delta x} \leq 1}$

if $\alpha = 1$ all fronts move from one cell boundary to the next and error is zero because there is no averaging