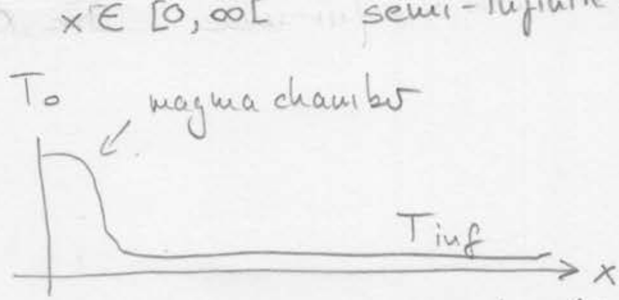
Lateral heat conduction

We are interested in the following problem:

PDE: $\frac{\partial T}{\partial t} - D \frac{1}{x^{d-1}} \frac{\partial}{\partial x}\left(x^{d-1} \frac{\partial T}{\partial x}\right) = 0 \quad x \in [0, \infty[$ semi-infinite domain

BC: $\left.\frac{\partial T}{\partial x}\right|_0 = 0$ "natural BC"

IC: $T_0(x)$



For the Occator problem the cylindrical coordinates $(d=2)$ are most relevant, but we'll keep it general.

How many parameters? $\rho, c_p, \kappa, T_{inf}$ + IC

If we want to non-dimensionalize, we lack obvious external scales!

Example: no clear length scale because domain is infinite

Basic Insight:
Assuming that heat is initially localized, the long-term evolution only depends on energy in the system!

Integral constraint: $\boxed{\int_0^\infty \rho c_p T_0 \, x^{d-1} \, dx = E}$

$\Rightarrow$ the length scale of the IC is not very informative
similarly the initial T-scale

Self-similar solution to heat equation

While there is no external length scale there is an internal diffusive length scale $\sqrt{Dt}$!

Is $x' = \frac{x}{\sqrt{Dt}}$ an appropriate dimensionless distance?

No, because t is not a parameter, it is an independent variable itself

$\Rightarrow$ new independent variable $\eta \sim \frac{x}{\sqrt{t}}$

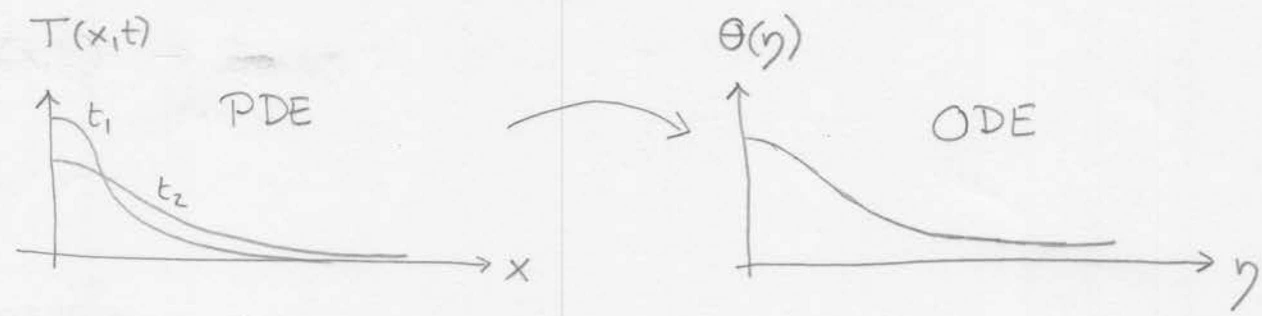
Using this hint we define: $\eta = b t^{\beta} x$ and $\theta(\eta) = a t^{\alpha} T(x,t)$

here η is the new dim. less independent variable

θ is the new dim. less dependent variable

a, b are constant choosen to make η and θ dimensionless.

α, β are exponents, from above argument we expect $\beta = -\frac{1}{2}$



New variables η, θ, are combinations of T, x, t hence we have one variable less: PDE $\rightarrow$ ODE

Indicates that T-profile at late time is essentially always the same $\rightarrow$ self-similar

Need to determine self similar ODE

The coefficients a, b and powers α, β need to be chosen to reduce PDE to ODE and make it dim-less

Transform variables:

$$\eta = b t^{\beta} x \;\rightarrow\; \frac{\partial \eta}{\partial t} = \beta\, b\, t^{\beta-1} x = \beta \frac{b t^{\beta} x}{t} = \beta \frac{\eta}{t}$$

$$\frac{\partial \eta}{\partial x} = b t^{\beta}$$

$$\theta(\eta) = a t^{\alpha}\, T(x,t) \;\rightarrow\; T = \frac{\theta}{a t^{\alpha}}$$

$$\frac{\partial}{\partial t} = \frac{\partial \eta}{\partial t}\frac{d}{d\eta} = \beta \frac{\eta}{t}\frac{d}{d\eta}$$

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{d}{d\eta} = b t^{\beta}\frac{d}{d\eta}$$

$$\frac{\partial T}{\partial t} = \beta \frac{\eta}{t}\frac{d}{d\eta}\left(\frac{\theta}{a t^{\alpha}}\right) = \frac{\beta\,\eta}{a t^{\alpha} t}\,\frac{d\theta}{d\eta}$$

$$\frac{\partial T}{\partial x} = b t^{\beta}\frac{d}{d\eta}\left(\frac{\theta}{a t^{\alpha}}\right) = \frac{b t^{\beta}}{a t^{\alpha}}\,\frac{d\theta}{d\eta}$$

$$\frac{\partial T}{\partial t} - D\,\frac{1}{x^{d-1}}\frac{\partial}{\partial x}\left(x^{d-1}\frac{\partial T}{\partial x}\right) \qquad x = \frac{\eta}{b t^{\beta}}$$

$$\frac{\beta\eta}{t}\,\cancel{\frac{1}{a t^{\alpha}}}\,\frac{d\theta}{d\eta} - D\,\frac{\cancel{(b t^{\beta})^{d-1}}}{\eta^{d-1}}\, b t^{\beta}\frac{d}{d\eta}\left[\frac{\eta^{d-1}}{\cancel{(b t^{\beta})^{d-1}}}\,\frac{b t^{\beta}}{\cancel{a t^{\alpha}}}\,\frac{d\theta}{d\eta}\right] = 0$$

$$\frac{\beta\eta}{t}\,\frac{d\theta}{d\eta} - D(b t^{\beta})^{2}\,\frac{1}{\eta^{d-1}}\frac{d}{d\eta}\left[\eta^{d-1}\frac{d\theta}{d\eta}\right] = 0$$

$$\beta\,\eta\,\frac{d\theta}{d\eta} - D\, b^{2}\, t^{2\beta+1}\,\frac{1}{\eta^{d-1}}\frac{d}{d\eta}\left[\eta^{d-1}\frac{d\theta}{d\eta}\right] = 0$$

The ODE cannot depend on $t \Rightarrow 2\beta + 1 = 0 \Rightarrow \boxed{\beta = -\tfrac{1}{2}}$ ✓

$$-\frac{\eta}{2}\,\frac{d\theta}{d\eta} - D b^{2}\,\frac{1}{\eta^{d-1}}\frac{d}{d\eta}\left[\eta^{d-1}\frac{d\theta}{d\eta}\right] = 0$$

multiply by 4 !

$$2\eta\,\frac{d\theta}{d\eta} + 4 D b^{2}\,\frac{1}{\eta^{d-1}}\frac{d}{d\eta}\left[\eta^{d-1}\frac{d\theta}{d\eta}\right] = 0 \qquad b = \frac{1}{\sqrt{4D}}$$

Independent similarity variable: $\eta = b t^{\beta} x \Rightarrow \boxed{\eta = \frac{x}{\sqrt{4Dt}}}$

self-similar ODE : $\boxed{2\eta \frac{d\theta}{d\eta} + \frac{1}{\eta^{d-1}} \frac{d}{d\eta}\left(\eta^{d-1} \frac{d\theta}{d\eta}\right) = 0}$

Note that η is independent of, d, dimension ! → Boltzmann variable

Still need to determine a, α and hence form of θ

Use integral constraint: $\int_0^\infty \rho c_p T x^{d-1} dx = E$

$\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4Dt}}$ $\quad dx = \sqrt{4Dt}\, d\eta$ $\quad$ if $x = 0$ then $\eta = 0$ for all $t < \infty$
if $x \to \infty$ then $\eta \to \infty$ for all $t < \infty$

$$\rho c_p \int_0^\infty T x^{d-1} dx = \rho c_p \int_0^\infty \frac{\theta}{a t^{\alpha}} \left(\sqrt{4Dt}\, \eta\right)^{d-1} \sqrt{4Dt}\, d\eta = E$$

$$\int_0^\infty \theta \frac{(4Dt)^{\frac{d}{2}}}{a t^{\alpha}} \eta^{d-1} d\eta = \frac{E}{\rho c_p} \qquad \int_0^\infty \theta\, t^{\frac{d}{2}-\alpha} \eta^{d-1} d\eta = \frac{E a}{\rho c_p (4D)^{d/2}} = 1$$

Again the t-term has to cancel. $\Rightarrow \frac{d}{2} - \alpha = 0 \Rightarrow \alpha = \frac{d}{2}$

also $\quad \alpha = \frac{\rho c_p (4D)^{d/2}}{E}$ $\quad$ so that $\quad \theta(\eta) = a t^{\alpha} T(x,t)$

$$\boxed{\theta(\eta) = \frac{\rho c_p}{E} (4Dt)^{\frac{d}{2}} \cdot T(x,t)}$$

Suppose we have solved self-similar ODE and have $\theta(\eta)$
↑
Gaussian !

then: $\boxed{T(x t) = \frac{E}{\rho c_p} \frac{1}{(4Dt)^{d/2}} \,\theta\left(\frac{x}{\sqrt{4Dt}}\right)}$

⇒ heat propagates as $x \sim \sqrt{t}$ irrespective of dimension

heat decays as $T \sim \frac{1}{t^{\frac{d}{2}}}$ ⇒ faster in higher dimensions
only for $d=1 \to T \sim \frac{1}{\sqrt{t}}$

⇒ test numerically