

## Melt migration - Fundamental analytic solutions

Dimensionless analytic solutions (dropping subscript  $D$ )

$$1) -\nabla \cdot [\phi^n \nabla h] + \phi^m h = \phi^m z + \Gamma$$

$$2) -\nabla^2 u = \phi^m (h - z)$$

$$3) \frac{\partial \phi}{\partial t} + \phi_c \nabla \cdot [\underline{v} \phi] = \phi^m (h - z) + \partial_a \Gamma$$

constitutive laws  $q = -\phi^n \nabla h$       $\underline{v} = -\nabla u$

$$p = h - z = \phi^m \nabla \cdot \underline{v}$$

## Steady Exchange Flow without melting

"Exchange flow" means that ice & melt move in

opposite directions: ice  $\uparrow$  & melt  $\downarrow$

no melting:  $\Gamma$

steady state:  $\phi = \text{const} = 1$

from equ 3:  $\phi_c \nabla \cdot \underline{v} = (h - z) = p$

substituting compaction relation

$$\phi_c p = p \Rightarrow p = 0 \Rightarrow \underline{v} = \text{const}$$

$\Rightarrow$  Equ 3 is trivially satisfied

Eqn 1 reduces to

$$-\nabla \cdot [\cancel{\phi'} \nabla h] + \cancel{\phi'} \underbrace{(h-z)}_p = 0$$
$$-\nabla^2 h = 0$$

Eqn 2 reduces to  $-\nabla^2 u = 0$

Integrating twice:  $h = a_1 z + a_2$

$$u = b_1 z + b_2$$

linear

Typically we determine coefficients from BC's but here we imagine an infinite vertical domain.

Instead we know from Eqn 3

$$p = h - z = 0 \Rightarrow a_1 z + a_2 - z = 0 \Rightarrow a_1 = 1 \quad a_2 = 0$$

$$\boxed{h = z}$$

From Darcy's law:  $q = -\cancel{\phi'} \frac{dh}{dz} = -1$

$$\boxed{q = -1}$$

From continuity:

$$\nabla \cdot [q_r + \underline{v}_s] = 0 \quad q = \frac{q_r}{K_c} \quad \underline{v} = \frac{\underline{v}_s}{K_c}$$

$$\nabla \cdot [q + \underline{v}] = 0 \quad (\text{dim.less})$$

$$\frac{d}{dz} [q + v] = 0 \Rightarrow q + v = c = \text{const.}$$

This constant is also typically determined from BC's and sets the "net motion" in an exchange flow the net motion is zero,  $c=0$ .

$$\Rightarrow \boxed{v = -q = 1}$$

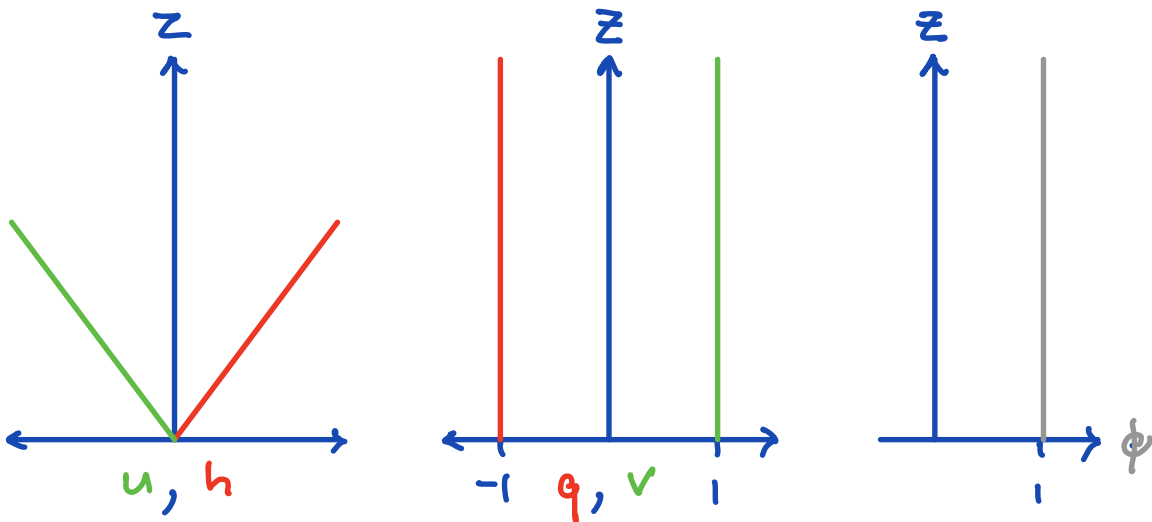
Finally, the velocity potential is given by

$$v = -\frac{du}{dz} \quad \text{where } v=1 \quad \text{and } u = b_1 z + b_2$$

$$1 = -b_1 \Rightarrow u = -z + c_1$$

here  $c_1$  is arbitrary as only the gradient matters,  $c_1 = 0$ .

$$\boxed{u = -z}$$



dimensionless solution:

$$\phi = 1, \quad h = z, \quad u = -z, \quad q = -1, \quad v = 1$$

This does not mean that ice and melt move at opposite & equal velocities!  $\nabla$

$q$  is a flux and  $v$  is a velocity.

Redimensionalize:

$$q_r = q_c q_D = -K_c = -K_0 \phi_c^n, \quad v_s = K_0 \phi_c^n$$

from def. of relative flux:  $q_r = \phi (v_f - v_s)$

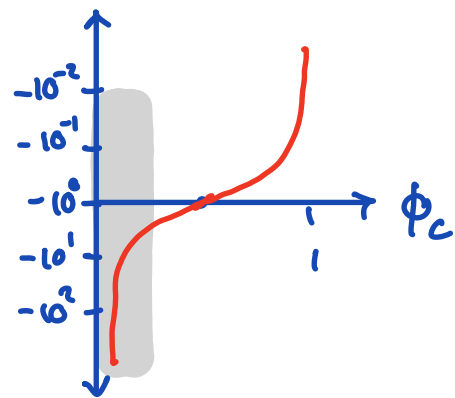
substituting:  $-K_c = \phi_c (v_f - K_c)$

$$-\frac{K_c}{\phi_c} = v_f - K_c$$

$$v_f = K_c - \frac{K_c}{\phi} = K_c \left(1 - \frac{1}{\phi}\right)$$

$$v_f = K_c \left(\frac{\phi-1}{\phi}\right) \quad K_c = v_s$$

$$\boxed{\frac{v_f}{v_s} = -\frac{1-\phi}{\phi}}$$



At low porosities  $|v_f| \gg |v_s|$ !

In other words, the solid velocities induced by melt migration are typically small.

## Instantaneous Compacting Column

Consider a vertical column of dimensionless height  $H_D = \frac{H}{x_c}$  with constant porosity  $\phi_c$  ( $\phi_p = 1$ ) and solid top and bottom boundaries ( $v_s = q_r = 0$ )

Compute the instantaneous flow problem ( $h, u, q, v$ ).

The dimensionless form of eqns 1 & 2 is:

$$\begin{aligned} 1) \quad & -\frac{d^2 h}{dz^2} + h = z && \text{with } q = -\frac{dh}{dz}\Big|_0 = -\frac{dh}{dz}\Big|_H = 0 \\ 2) \quad & -\frac{d^2 u}{dz^2} = h - z && \text{with } v = -\frac{du}{dz}\Big|_0 = -\frac{du}{dz}\Big|_H = 0 \end{aligned}$$

Equation 1 is non-homog. 2<sup>nd</sup> order ODE with constant coefficients. Solve by method of undetermined coefficients.

$$h = h_h + h_p \quad h_h = c_1 e^{r_1 z} + c_2 e^{r_2 z}$$

$$h_p = c_3 z$$

substitute  $e^{r_1 z}$  into homogeneous solution

$$-r^2 e^{rz} + e^{rz} = 0 \Rightarrow r^2 = 1 \quad r = \pm 1$$

homogeneous solution:  $h_h = c_1 e^z + c_2 e^{-z}$

substitute  $h_p$  into non-hom. equ

$$-\cancel{\frac{d^2 h_p}{dz^2}} + h_p = z \Rightarrow c_3 = 1 \quad h_p = z$$

Full solution:  $h = c_1 e^z + c_2 e^{-z} + z$

$$\frac{dh}{dz} = c_1 e^z - c_2 e^{-z} + 1$$

$$\text{BC: } \left. \frac{dh}{dz} \right|_0 = c_1 - c_2 + 1 = 0 \Rightarrow c_2 = c_1 + 1$$

$$\left. \frac{dh}{dz} \right|_H = c_1 e^H - c_2 e^{-H} + 1 = 0$$

$$\text{solving for } c_1 \text{ \& } c_2: c_1 = \frac{e^{-H} - 1}{e^H - e^{-H}} \quad c_2 = \frac{e^H - 1}{e^H - e^{-H}}$$

Hence the solution is:

$$h(z) = z + \frac{e^{-H} - 1}{e^H - e^{-H}} e^z + \frac{e^H - 1}{e^H - e^{-H}} e^{-z}$$

$$q(z) = -\frac{dh}{dz} = -1 - \frac{e^{-H} - 1}{e^H - e^{-H}} e^z + \frac{e^H - 1}{e^H - e^{-H}} e^{-z}$$

↑  
main  
solution

boundary  
layers