Numerical solution of melt migration

PDE system

1) $-\nabla \cdot [\phi_D^n \nabla h_D] + \phi_D^m h_D = \phi_D^m z_D$

2) $-\nabla^2 u_D = \phi_D^m (h_D - z_D) = \phi_D^m p_D$

3) $\frac{\partial \phi_D}{\partial t_D} + \phi_c \nabla \cdot [\underline{v}_D \phi_D] = \phi_D^m (h_D - z_D) = \phi_D^m p_D$

Write 3 new functions:

1) solve_helmholtz.m → hD pD qD
2) solve_poisson.m → uD vD
3) solve_melt_transport.m → phiD
4) comp_flux_helm.m

3 sets of BC.

1) BC.h → Helmholtz equ
2) BC.u → Poisson equ
3) BC.phi → Transport equ

⇒ 1D solution

Solution of Helmholtz Eqn

new equation

Continuous: $-\nabla \cdot [\phi_D^n \nabla h_D] + \phi_D^m h_D = \phi_D^m z_D$

Discrete: $\underbrace{\left(-\underline{\underline{D}}\ \underline{\underline{\text{Phi\_n}}}\ \underline{\underline{G}} + \underline{\underline{\text{Phi\_m}}}\right)}_{\underline{\underline{L}}} \underline{hD} = \underbrace{\underline{\underline{\text{Phi\_m}}}\ xc}_{\underline{fs}}$

Porosity matrices:

1) Hydraulic conductivity term:

$\underline{\underline{\text{Phi\_n}}}$ on faces like $\underline{\underline{\text{Kd}}} \Rightarrow$ Nf by Nf

$\underline{\underline{\text{Phi\_n}}}$ = comp_mean($\underline{\text{phi}}$.^n, $\underline{\underline{M}}$, -1, Grid, 1) ;

2) Viscosity term:

$\underline{\underline{\text{Phi\_m}}}$ in cell centers $\Rightarrow$ N by N

$\underline{\underline{\text{Phi\_m}}}$ = spdiags($\underline{\text{phi}}$.^m, 0, Grid.N, Grid.N)

Solve: $\underline{\underline{L}}\ \underline{\underline{hD}} = \underline{fs}$

$\underline{pD} = \underline{hD} - \underline{xc}$

$\underline{qD}$ = comp_flux_helmholtz ...

Solution of Poisson Eqn.

solve this equ before

Continuous: $-\nabla^2 u_D = \phi_D^m p_D$

Discrete: $\underbrace{-\underline{\underline{D}}\,\underline{\underline{G}}}_{\underline{\underline{L}}}\ \underline{\underline{uD}} = \underbrace{\underline{\underline{Phi\_m}}\ \underline{\underline{pD}}}_{\underline{fs}}$

Solve: $\underline{\underline{L}}\ \underline{uD} = \underline{fs}$

$\underline{vD}$ = comp_flux ...

Solution of Transport Eqn.

Continuous: $\frac{\partial \phi_D}{\partial t_D} + \phi_c \nabla \cdot [\underline{v}_D \phi_D] = \phi_D^m p_D$

Note: for $m \neq 1$ or $m \neq 0 \Rightarrow$ non-linear eqn

assume $m = 0$ or $m = 1$!

Rearrange:

$m=1$: $\frac{\partial \phi_D}{\partial t} + \phi_c \nabla \cdot [\underline{v}_D \phi_D] - p_D \phi_D = 0$

$m=0$ $\frac{\partial \phi_D}{\partial t} + \phi_c \nabla \cdot [\underline{v}_D \phi_D] = 0$

both: $\frac{\partial \phi_D}{\partial t} + \phi_c \nabla \cdot [\underline{v}_D \phi_D] - m\, p_D \phi_D = 0$

as before: $\underline{pD} .* \underline{phiD} = \underline{\underline{PD}} * \underline{phiD}$

where $\underline{\underline{PD}} = \text{spdiags}(\underline{\underline{pD}}, 0, N, N);$

Discrete:

$$\underline{phiD}^{n+1} - \underline{phiD}^{n} + \Delta t\, \phi_c \underbrace{\left[\underline{\underline{D}}\, \underline{\underline{A}}(\underline{\underline{vD}}) - m\, \underline{\underline{PD}}\right]}_{\underline{\underline{L}}} \left(\theta\, \underline{phiD}^{n} + (1-\theta)\, \underline{phiD}^{n+1}\right) = \Delta t\, \underline{fs}$$

collect "n+1" on r.h.s. and "n" on l.h.s.

$$\underbrace{\left[\underline{\underline{I}} + \Delta t\, \phi_c (1-\theta)\, \underline{\underline{L}}\right]}_{\underline{\underline{IM}}} \underline{phiD}^{n+1} = \underbrace{\left[\underline{\underline{I}} - \Delta t\, \theta\, \underline{\underline{L}}\right]}_{\underline{\underline{EX}}} \underline{phiD}^{n} + \Delta t\, \underline{fs}$$

solve: $\underline{\underline{IM}}\ \underline{phiD}^{n+1} = \underline{\underline{EX}}\ \underline{phiD}^{n} + \Delta t\, \underline{fs}$

Time evolution:

```
for n = 1:Nt
    [hD, pD, qD] = solve_Helmholtz...
    [uD, vD] = solve_Poisson...
    phiD = solve_melt_transport...
end
```

Compute Darcy flux

Continuous: $q_D = -\Phi_D^n \nabla h_D$

Discrete: $\underline{q}D = -\underline{\underline{\text{Phi\_n}}}\,\underline{\underline{G}}\,\underline{h}D$ o.k. interior

need to reconstruct flux on boundary.

Remember comp_flux.m

interior:

$\underline{q} = -\underline{\underline{Kd}}\,\underline{\underline{G}}\,\underline{h}$;

$\underline{res} = \underline{\underline{D}}\,\underline{q} - \underline{fs}$;

boundary:

$\underline{q} = \underline{\text{sign}} * \underline{res} \frac{V}{A}$

For the Helmholtz eqn

$\underline{q}D = -\underline{\underline{\text{Phi\_n}}}\,\underline{\underline{G}}\,\underline{h}D$

$\underline{res} = \underline{\underline{D}}\,\underline{q}D + \underline{\underline{\text{Phi\_m}}}\,\underline{h}D - \underline{fs}$

$\Rightarrow$ comp_flux_helm.m