Time integration

heat equation $\rightarrow$ transient linear PDE

$$\rho c_p \frac{\partial T}{\partial t} \underbrace{- \nabla \cdot [\kappa \nabla T]}_{\underline{\underline{L}} = -\underline{\underline{D}}\,\underline{\underline{Kd}}\,\underline{\underline{G}}} = f_s$$

$\Rightarrow$ just need to discretize the time derivative

$T(\underline{x}) = \underline{u}$ because $\underline{T}$ looks like a matrix

and $\underline{t}$ looks like time!

$$\boxed{\frac{\partial T}{\partial t} \approx \frac{\underline{u}^{n+1} - \underline{u}^{n}}{\Delta t}}$$ where $\Delta t = t^{n+1} - t^{n}$

n = time level $t^0 = 0$

$\underline{\underline{RhoCp}} = \rho\, c_p\, \underline{\underline{I}}$ if ρ = const. c_p = const.

$=$ spdiags($\underline{\text{rho}}$.*$\underline{\text{cp}}$, 0, Nx, Nx);

Nx·1 vectors

substitute

$$\underline{\underline{RhoCp}}\,(\underline{u}^{n+1} - \underline{u}^{n}) + \Delta t\; \underline{\underline{L}}\; \underline{u}^{?} = \Delta t\; \underline{f_s}$$

Q: At what time-level do we evaluate $\underline{\underline{L}}\,\underline{u}$?

Theta-method

$$\underline{u}^{\theta} = \theta\, \underline{u}^{n} + (1-\theta)\, \underline{u}^{n+1} \qquad \theta \in [0,1]$$

subsitute

$$\underline{\underline{RhoCp}}\,(\underline{u}^{n+1} - \underline{u}^{n}) + \Delta t\, \underline{\underline{L}}\, \underline{u}^{\theta} = \Delta t\, \underline{f_s}$$

$$\underline{\underline{RhoCp}}\,(u^{n+1} - \underline{u}^{n}) + \Delta t\, L \left[\theta\, \underline{u}^{n} + (1-\theta)\, \underline{u}^{n+1}\right] = \Delta t\, \underline{f_s}$$

Note: $\underline{u}^{n}$ is known $\qquad \underline{u}^{n+1}$ is unknown

$\Rightarrow$ collect $\underline{u}^{n+1}$ on l.h.s. and $\underline{u}^{n}$ on r.h.s.

$$\underbrace{\left[\underline{\underline{RhoCp}} + \Delta t\,(1-\theta)\, \underline{\underline{L}}\right]}_{\underline{\underline{IM}}} \underline{u}^{n+1} = \Delta t\, \underline{f_s} + \underbrace{\left[\underline{\underline{RhoCp}} - \Delta t\, \theta\, \underline{\underline{L}}\right]}_{\underline{\underline{EX}}} \underline{u}^{n}$$

Linear system for a time step:

$$\boxed{\underline{\underline{IM}}\, \underline{u}^{n+1} = \Delta t\, \underline{f_s} + \underline{\underline{EX}}\, \underline{u}^{n}}$$

Implicit matrix: $\underline{\underline{IM}} = \underline{\underline{RhoCp}} + \Delta t\,(1-\theta)\, \underline{\underline{L}}$

Explicit matrix: $\underline{\underline{EX}} = \underline{\underline{RhoCp}} - \Delta t\, \theta\, \underline{\underline{L}}$

Properties of Theta Method

$\Rightarrow$ combines different methods

For $\theta = 1$: Forward Euler Method

$\underline{\underline{M}} = \underline{\underline{RhoCp}}$ (diagonal)

$\underline{u}^{n+1} = \underline{\underline{RhoCp}}^{-1} \left(\Delta t \, \underline{fs} + \underline{\underline{EX}} \, \underline{u}^n \right)$

diagonal is easy to invert

$$\underline{\underline{RhoCp}}^{-1} = \begin{bmatrix} \ddots & & \\ & \frac{1}{\rho c_p} & \\ & & \ddots \end{bmatrix}$$

- explicit expression for $\underline{u}^{n+1}$
  don't need to solve linear system
- computationally cheap
- first-order accurate
- conditionally stable $\Delta t \leq \frac{\Delta x^2}{2D}$

$$D = \frac{k}{\rho c_p}$$

For $\theta = 0$: Backward Euler method

$$\underline{\underline{EX}} = \underline{\underline{RhoCp}} \quad \text{(diagonal)}$$

$$\underline{\underline{\mathbb{M}}}\, \underline{u}^{n+1} = \Delta t\, \underline{fs} + \underline{\underline{EX}}\, u^n$$

- solve a linear system
  $\Rightarrow$ implicit method
- computationally expensive
- first-order accurate
- unconditionally stable

For $\theta = \frac{1}{2}$: Crank-Nicholson method

$$\underline{\underline{\mathbb{M}}}\, \underline{u}^{n+1} = \Delta t\, \underline{fs} + \underline{\underline{EX}}\, u^n$$

- solve linear system
  $\Rightarrow$ implicit method
- second-order accurate
- unconditionally stable (oscillation limit)